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THE LOCAL BOUNDEDNESS OF THE PERTURBED MOTIONS OF A GYROSCOPE IN GIMBALS WITH DISSIPATIVE AND ACCELERATING FORCES*

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The motion of an unbalanced gyroscope in gimbals in a central Newtonian field of forces is considered, taking the masses of the suspension rings into account. It is assumed that there is a moment of forces of viscous friction acting on the axis of rotation of one of the rings, and there is an accelerating (electromagnetic) moment applied to the axis of rotation axis of the other ring. The equations of motion have a partial solution such that the mean velocity of the outer ring is perpendicular to the direction from the centre of gravitation S to the stationary point O, the middle plane of the inner ring contains this direction, and the gyroscope rotates about SO with an arbitrary constant angular velocity.

The equations of perturbed motions of the system in the neighbourhood of the corresponding state of equilibrium are obtained to within terms of order three. The characteristic equation of the system is considered and the coefficients of the equation are found in the region F_0 of admissible values of the parameters. The question of the distribution of eigenvalues with respect to the imaginary axis is studied. A region in F_0 is constructed in which the pairs of complex conjugate eigenvalues have small real parts among which there are some positive ones, and the absolute values of the resonance mistuning between the imaginary parts are not small. In this region we obtain sufficient conditions for local uniform boundedness of perturbed motions of the gyroscope in gimbals with dissipative and accelerating forces with respect to the partial solution mentioned above. These conditions are found in the form of constraints for the coefficients of the normal form and, eventually, for the original parameters of the system and the real and imaginary parts of the eigenvalues. To provide illustrative interpretation, some special cases are considered and the regions of

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local uniform boundedness within flat regions of admissible values of the parameters are constructed using a computer.

1. We shall consider the motion of a massive dynamically symmetrical gyroscope in gimbals in a Newtonian field of forces with centre S, taking into account the masses of the suspension rings. The axes of rotation of the outer ring (the frame) and the inner ring (the mantle) are perpendicular to one another, and so are the axes of the gyroscope (the rotor) and the inner ring. The directions in question intersect each other at the stationary point O of the gyroscope. Each of the suspension rings rotates about one of its main axes of inertia. The gyroscope rotates about the axis of dynamical symmetry, which coincides with the main axis of inertia of the inner ring. Let there be a moment of forces of viscous friction acting on the axis of the outer (or the inner) ring, and let there be an electromagnetic device on the axis of the inner (or the outer) ring which produces an accelerating moment ating in the same direction as the direction of rotation of the ring and proportional to the angular velocity of the ring (/1/, p.182). We assume that there is no moment of friction forces about the axis of rotation of the gyroscope or that it is cancelled by an electromagnetic moment applied to the rotor (/2/, p.85). Let the centre of mass of the outer ring lie on the stationary axis of rotation of the outer ring, which is perpendicular to SO, and let the centre of mass P of the system consisting of the inner ring and the gyroscope lie on the axis of symmetry of the rotor.

We will introduce into the consideration two orthogonal right-handed coordinate systems, namely, the system $OXYZ$ attached to the stationary platform in such a way that the X-axis passes through the centre of gravitation S and the Z-axis is directed along the stationary axis of inertia of the outer ring, and the system $O\xi\eta\zeta$ attached to the inner ring with axes parallel to the main axes of inertia of the inner ring, so that ξ and ζ are measured along the axes of rotation of the inner ring and the rotor, respectively. We will assume that initial state of the system is such that the middle planes of the suspension rings coincide and contain the direction SO. The current position of the system in question with respect to the stationary platform will be determined by the Euler angles ψ, θ and φ, that is, the angles of rotation of the outer ring, the inner ring, and the rotor, respectively.

The equations of motion of the gyroscope in gimbals with dissipative and accelerating forces acting on the axes of the suspension rings have the form

$$\mathbf{p}^{\cdot} = -\partial H/\partial q - \mathbf{F}\partial H/\partial \mathbf{p}, \ \mathbf{q}^{\cdot} = \partial H/\partial \mathbf{p} \tag{1.1}$$

$$H = {}^1/_2 \{(A + A_1)^{-1}p_\theta^2 + ((A + B_1)\sin^2\theta + C_1\cos^2\theta +$$
$$A_2)^{-1} \times (p_\psi - p_\varphi \cos\theta)^2 + C^{-1}p_\varphi^2\} + Mg\zeta_0 \sin\psi\sin\theta +$$
$${}^3/_2 gR^{-1}(A_1\cos^2\psi + B_1\sin^2\psi\cos^2\theta + (C + C_1 - A)\sin^2\psi\sin^2\theta)$$
$$\mathbf{F} = \operatorname{diag}(k_\psi, \ k_\theta, \ 0)$$
$$F = {}^1/_2 k_\psi (\partial H/\partial p_\psi)^2 + {}^1/_2 k_\theta \ \partial H/\partial p_\theta)^2$$

Here $\mathbf{p} = (p_\psi, p_\theta, p_\varphi)^T$ are the generalized momenta corresponding to the coordinates $\mathbf{q} = (\psi, \theta, \varphi)^T$, H is the Hamilton function, F is the matrix of Rayleigh functions F, A and C are the equatorial and polar moments of inertia of the gyroscope with respect to O; A_1, B_1 and C_1 are the main moments of inertia of the inner ring with respect to O; A_2 is the moment of inertia of the outer ring with respect to the Z axis, M is the mass of the system consisting of the inner ring and the rotor, g is the acceleration due to gravity forces at the distance R from S to O, ζ_0 is the ζ coordinate of the centre of mass P, $k_\psi > 0$ (or $k_\theta > 0$) is the coefficient of viscous friction acting on the axis of the outer ring (or the inner ring) and $k_\theta < 0$ (or $k_\psi < 0$). Thus, $|k_\theta|$ (or $|k_\psi|$) is the steepness of the characteristic of the electromagnetic device on the axis of the inner (or the outer) ring producing the acceleration moment and $k_\psi k_\theta < 0$.

Since the angle of rotation φ of the gyroscope does not appear explicitly in the Hamiltonian H and the forces acting around the axis of the gyroscope cancel each other, system (1.1) admits of the integral of motion $p_\varphi = \mathrm{const}$, and a reduced system with two degrees of freedom can be extracted from Eqs.(1.1).

2. Eqs.(1.1) have the partial solution

$$p_\psi = 0, \ p_\theta = 0, \ p_\varphi = C\omega' \tag{2.1}$$
$$\psi = \pi/2, \ \theta = \pi/2, \ \varphi = \omega' t + \varphi_0$$

in which case the middle plane of the outer ring is perpendicular to SO, the middle plane of the inner ring contains SO, and the gyroscope rotates with an arbitrary constant angular velocity ω' about SO.

We shall obtain the equations of perturbed motions of the reduced system in the vicinity of the state of equilibrium

$$p_\psi = p_\theta = 0,\ \psi = \theta = \pi/2 \tag{2.2}$$

which corresponds to the stationary motions (2.1) of the original system (1.1). We set

$$p_\psi = p_1',\ p_\theta = p_2',\ \psi = \pi/2 + q_1',\ \theta = \pi/2 + q_2'$$

and we find the expansion of the Hamilton function of the reduced system in the vicinity of the state of equilibrium to within terms of order four with respect to the perturbations p_m', q_m' $(m = 1, 2)$. We introduce the new dimensionless variables p_m, q_m, the time τ, the angular velocity ω, the coordinates k_m, and the parameters $a, a_1, b_1, c_1, a_2, e, \delta$ defined by the formulae

$$\begin{gathered} p_m' = \pi_* p_m,\ q_m' = q_m\ (m = 1, 2),\ t = \sigma_* \tau \\ \omega' = \frac{\omega}{\sigma_*},\quad k_\psi = \pi_* k_1,\quad k_\theta = \pi_* k_2,\quad a = \frac{A}{C} \\ a_1 = \frac{A_1}{C},\quad b_1 = \frac{B_1}{C},\quad c_1 = \frac{C_1}{C},\quad a_2 = \frac{A_2}{C},\quad e = \operatorname{sign} \zeta_0 \\ \delta = \frac{C}{RM|\zeta_0|},\quad \pi_* = (CMg|\zeta_0|)^{1/2},\quad \sigma_* = \left(\frac{C}{Mg|\zeta_0|}\right)^{1/2} \end{gathered} \tag{2.3}$$

We obtain an expansion of the Hamiltonian of the reduced system in the form

$$\begin{gathered} H = H_2 + H_4 + \ldots \\ H_n = \sum_{|\nu|=n} h_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1} p_2^{\nu_2} q_1^{\nu_3} q_2^{\nu_4} \quad (n = 2, 4, \ldots) \end{gathered} \tag{2.4}$$

where $\nu_1, \ldots, \nu_4$ are non-negative integral numbers and $|\nu| = \nu_1 + \ldots + \nu_4$. The following coefficients $h_{\nu_1\nu_2\nu_3\nu_4}$ of the forms H_2 and H_4 are non-zero:

$$\begin{gathered} 2h_{2000} = e_1,\ h_{1001} = \omega e_1,\ 2h_{0200} = e_2 \\ 2h_{0020} = h_1,\ 2h_{0002} = \omega^2 e_1 + h_2 \\ 2h_{2002} = (a + b_1 - c_1) e_1^2,\ 6h_{1003} = \omega(5a + 5b_1 - 6c_1 - \\ a_2) e_1^2,\ 2h_{0040} = -{}^1/_4 e - {}^1/_3 h_1,\ 2h_{0022} = \\ -{}^1/_2 e - h_2,\ 2h_{0004} = {}^1/_3 \omega^2 (2a + 2b_1 - 3c_1 - a_2) e_1^2 - \\ {}^1/_4 e - {}^1/_3 h_2 \end{gathered} \tag{2.5}$$

$$\begin{gathered} e_1 = (a + b_1 + a_2)^{-1},\ e_2 = (a + a_1)^{-1},\ h_1 = -e + \\ 3\delta(a - 1 + a_1 - c_1),\ h_2 = -e + 3\delta(a - 1 + b_1 - c_1) \end{gathered} \tag{2.6}$$

The equations of perturbed motions of the reduced system in the vicinity of the state of equilibrium (2.2) expressed in terms of the dimensionless variables defined by (2.3) have the form

$$\frac{dp_m}{d\tau} = -\frac{\partial H}{\partial q_m} - k_m \frac{\partial H}{\partial p_m},\qquad \frac{dq_m}{d\tau} = \frac{\partial H}{\partial p_m}\qquad (m = 1, 2) \tag{2.7}$$

Remarks. 2.1. The dimensionless parameter δ is small, since in the approximate discussion of the central Newtonian field of forces /3/ it is assumed that the distance R is much greater than the dimensions of the gyroscope in gimbals. The limiting case corresponds to a uniform field of gravitational forces.

2.2. For the stationary motions (2.1), if the centre of mass P lies closer to (further from) the centre of gravitation S than the stationary point O, then we find that $e = -1$ $(e = 1$, respectively).

3. We introduce into consideration the following domain of admissible values of the parameters: $F_0 = \{\mathbf{c} = (\omega, k_1, k_2, a, a_1, b_1, c_1, a_2, \delta)$: $k_1 > 0,\ k_2 < 0$ (or $k_1 < 0,\ k_2 > 0$), $a > {}^1/_2,\ a_1 + b_1 > c_1,\ b_1 + c_1 > a_1,\ c_1 + a_1 > b_1,\ a_2 > 0$, and δ is a small positive number).

The characteristic equation of system (2.7) has the form

$$\lambda^4 + P_1\lambda^3 + (P_2 + Q_2)\lambda^2 + P_3\lambda + Q_4 = 0 \tag{3.1}$$

Computing the coefficients of (3.1) in accordance with /4/, we get

$$\begin{gathered} P_1 = k_1 e_1 + k_2 e_2,\ P_2 = k_1 k_2 e_1 e_2,\ Q_2 = \omega^2 e_1 e_2 + \\ e_1 h_1 + e_2 h_2,\ P_3 = (k_1 h_2 + k_2 h_1) e_1 e_2,\ Q_4 = e_1 e_2 h_1 h_2 \end{gathered} \tag{3.2}$$

Substituting (2.6) into (3.2), we get the final expressions for the coefficients of the characteristic polynomial in terms of $\mathbf{c} \in F_0$ and $e = \pm 1$.

Suppose that Eq.(3.1) has two pairs of complex conjugate roots. It follows from the Vieta formulae that the condition $Q_4 > 0$ is necessary under the assumption made above. The condition is satisfied according to the corresponding expression (3.2). Remarks 2.1 and the notation (2.6). If

$$\Omega = P_1(P_2 + Q_2)P_3 - P_1^2Q_4 - P_3^2 \neq 0 \tag{3.3}$$

then the real parts of the pairs of complex conjugate eigenvalues are non-zero. We consider the following domains:

$$N_0 = \{\mathbf{c}: \mathbf{c} \in F_0,\ P_1 > 0,\ P_2 + Q_2 > 0,\ P_3 > 0,$$
$$\Omega > 0\},\ N_4 = \{\mathbf{c}: \mathbf{c} \in F_0,\ P_1 < 0,\ P_2 + Q_2 > 0,$$
$$P_3 < 0,\ \Omega > 0\},\ N_2 = F_0 \setminus \{\mathbf{c}: P_1P_3 \geqslant 0,$$
$$P_2 + Q_2 \geqslant 0,\ \Omega \geqslant 0\},\ e = \pm 1$$

The number Ω appearing in the definition of the above domains can be expressed in terms of the coefficients of the characteristic Eq.(3.1) by means of formula (3.3). Using the theorem on the distribution of the roots of polynomials in the half-planes $\operatorname{Re}\lambda < 0$ and $\operatorname{Re}\lambda > 0$ /5/ applied to the case $Q_4 > 0$ under consideration, we arrive at the following results.

Proposition. For all the roots of (3.1) and (3.2) to have negative real parts it is necessary and sufficient that $\mathbf{c} \in N_0$ (see /1/). The real parts of all eigenvalues are positive if and only if $\mathbf{c} \in N_4$. If $\mathbf{c} \in N_2$, then the characteristic equation has two roots in the half-plane $\operatorname{Re}\lambda < 0$ and two roots in the half-plane $\operatorname{Re}\lambda > 0$.

Remark. 3.1. From the above proposition, the definition of N_0, N_4 and N_2, expression (3.2) for P_3 Remark 2.1, and notation (2.6) it follows that in general, for any $\mathbf{c} \in F_0$, the distribution of eigenvalues with respect to the imaginary axis changes if the change $e = -1 \leftrightarrow e = 1$ occurs.

4. Let $\mathbf{c} \in N_0$. Then the uniform rotations (2.1) of the gyroscope in gimbals are asymptotically stable /6/ with respect to the variables p_ψ, p_θ, p_φ, ψ, θ under parametric perturbations of the construction parameters.

Henceforth we shall assume that $\mathbf{c} \in N_4 \cup N_2$. Then the stationary motions (2.1) are unstable /6/. In addition, let all the roots of the characteristic Eq.(3.1), (3.2) have small real parts. For the real parts of the eigenvalues to be small it is necessary that P_1 and P_3 given by (3.2) be small, and it is sufficient that P_1 and P_3 be small and $\mathbf{c} \in G$, where

$$G = \{\mathbf{c}: \mathbf{c} \in F_0,\ P_2 + Q_2 > 2\sqrt{Q_4}\},\ e = \pm 1$$

It is obvious that $G \cap N_4 \neq \varnothing$ and $G \cap N_2 \neq \varnothing$. We denote the roots of (3.1) and (3.2) by $\mu\alpha_m \pm i\beta_m$ $(m = 1, 2)$, where $\mu > 0$ is a small parameter, $\beta_m > 0$, and where one of the numbers α_m is positive if $\mathbf{c} \in N_2$ and both of them are positive if $\mathbf{c} \in N_4$. To fix our ideas we shall assume that $\beta_1 > \beta_2$.

We shall investigate the sufficient conditions for local boundedness /7, 8/ of the solutions of the equations of perturbed motions (2.7) with respect to the origin $p_m = q_m = 0$ $(m = 1, 2)$ of the coordinate system for small P_1 and P_3 and $\mathbf{c} \in G \cap (N_4 \cup N_2)$*. We remark that to study the problem of the local boundedness of perturbed motions, it is necessary to consider Eqs.(2.7) with terms which are non-linear with respect to perturbations. The analysis below represents the case where one can neglect terms of order higher than three in (2.7).

Remarks. 4.1. The coefficient P_3 defined by (3.2) is small if $|k_1 + k_2|$ is small (see Remark 2.1 and (2.6)). Then the coefficient P_1 defined by (3.2) is small provided $|b_1 + a_2 - a_1|$ is small (see (2.6)). Therefore, the assumption that the real parts of the eigenvalues of system (2.7) are small imposes the above restrictions on the construction parameters $\mathbf{c} \in F_0$ of the gyroscope in gimbals.

4.2. The cases where the pairs of complex conjugate eigenvalues have small real parts were referred to by Kamenkov /9/ as cases which are close to being critical. In these cases the notion of local uniform boundedness of the solutions of (2.7) corresponds to the notion of stability in the sense of Kamenkov /9, 10/ translated into a formal language. The notion of local boundedness is convenient because its definition involves the initial and current estimates δ and ε of the region appearing in the definition of stability in the sense of

*For the solutions of an autonomous system of the form (2.7), the definition of local uniform boundedness used in /7, 8/ was stated in: S.A. Belikov, Local boundedness of solutions of an autonomous system of order four with small positive real parts of the eigenvalues, Leningrad Institute of Aviation Instruments, Leningrad, 1987. Deposited in VINITI 26.03.87, 2206-B87.

Kamenkov.

5. To obtain sufficient conditions for local boundedness, we carry out certain transformations of system (2.7), whose coefficients depend on the choice of **c** and $e = \pm 1$. In the construction of the transformations we define the hypersurfaces

$$R_2 = \{\mathbf{c}: \mathbf{c} \in F_0,\ P_2 + Q_2 = 2\sqrt{Q_4}\}$$
$$R_4 = \{\mathbf{c}: \mathbf{c} \in F_0,\ 3(P_2 + Q_2) = 10\sqrt{Q_4}\},\ e = \pm 1$$

R_2 is the boundary of G. If $\mathbf{c} \in R_2$, then the absolute value of the resonance mistuning $\varepsilon_{1,1}(\mathbf{c}) = \beta_1(\mathbf{c}) - \beta_2(\mathbf{c})$ is small. Henceforth we shall exclude from consideration a small neighbourhood R_2^0 of the surface R_2 such that $(\forall \mathbf{c} \in G \setminus R_2^0)\ |\varepsilon_{1,1}(\mathbf{c})| > \sqrt{\mu}$.

Let $\mathbf{c} \in (G \setminus R_2^0) \cap (N_4 \cup N_2)$. The linear transformation $\mathbf{z} = \mathrm{S}\mathbf{x}$, $\mathbf{z} = (p_1, p_2, q_1, q_2)^T$, $\mathrm{S} = \|s_{kl}\|_{k,l=1}^4$ $\det \mathrm{S} \neq 0$, $\mathbf{x} = (x_1, x_2, x_3, x_4)$ reduces the matrix **B** of the linearized system (2.7) to the real Jordan form. Let us write down the elements of **S**. We set

$$r_1 = k_2 e_2 \mu\alpha_1 + (\mu\alpha_1)^2 - \beta_1^2 + e_2 h_2 \tag{5.1}$$
$$i_1 = k_2 e_2 \beta_1 + 2\mu\alpha_1\beta_1$$
$$g_1 = |\beta_1((r_1^2 + i_1^2) h_1 + ((\mu\alpha_1)^2 + \beta_1^2)^2 \omega^2 e_2)|^{-1/2}$$

Then

$$s_{11} = (-(k_1\mu\alpha_1 + h_1) r_1 + k_1\beta_1 i_1) g_1 \tag{5.2}$$
$$s_{21} = -((\mu\alpha_1)^3 - 3\mu\alpha_1\beta_1^2)\omega g_1,\quad s_{31} = (\mu\alpha_1 r_1 - \beta_1 i_1) g_1$$
$$s_{41} = -((\mu\alpha_1)^2 - \beta_1^2)\omega e_2 g_1$$
$$s_{13} = ((k_1\mu\alpha_1 + h_1) i_1 + k_1\beta_1 r_1) g_1$$
$$s_{23} = (3(\mu\alpha_1)^2\beta_1 - \beta_1^3)\omega g_1,\quad s_{33} = -(\mu\alpha_1 i_1 + \beta_1 r_1) g_1,$$
$$s_{43} = 2\mu\alpha_1\beta_1\omega e_2 g_1$$

The formulae for s_{k2}, s_{k4} $(k = 1, \ldots, 4)$ can be found from the expression for s_{k1}, s_{k3}, respectively, by the change $\alpha_1 \to \alpha_2$ and $\beta_1 \to -e\beta_2$ in (5.1) and (5.2). Substitution of (5.1) and (2.6) into (5.2) yields the final expressions for the elements of **S** in terms of $\mathbf{c} \in (G \setminus R_2^0) \cap (N_4 \cup N_2)$ and α_m, β_m $(m = 1, 2)$.

R_4 and G have a non-empty intersection. If $\mathbf{c} \in R_4$, then the absolute value of the resonance mistuning $\varepsilon_{1,3}(\mathbf{c}) = \beta_1(\mathbf{c}) - 3\beta_2(\mathbf{c})$ is small. We exclude from consideration a small neighbourhood R_4^0 of R_4 such that $(\forall \mathbf{c} \in G \setminus R_4^0)\ |\varepsilon_{1,3}(\mathbf{c})| > \sqrt{\mu}$.

Let

$$\mathbf{c} \in (G \setminus (R_2^0 \cup R_4^0)) \cap (N_4 \cup N_2) \tag{5.3}$$

As a result of the normalization transformation $\mathbf{x} \to \mathbf{y}$ constructed with the use of a modified version of the Depri-Horikemel method, which is analogous to the Mersmann modification /10/, and the introduction of the polar coordinates

$$y_m = \rho_m \cos\vartheta_m,\ y_{m+2} = \rho_m \sin\vartheta_m\ (m = 1, 2)$$

the system of ordinary differential equations in **x** obtained from (2.7) by the change of variables $\mathbf{z} = \mathrm{S}\mathbf{x}$ can be reduced to a normal form at $\mu = 0$ which is continuous with respect to μ. The equations determining ρ_m becomes separated. In the cases where the problem of local boundedness of the solutions of (2.7) with respect to the origin of the coordinate system can be solved by means of terms of order not greater than three independently of any forms of higher orders, there is no need to consider the equations determining ϑ_m, since the problems of local boundedness in **z**, **x**, **y**, or ρ_m $(m = 1, 2)$ are equivalent.

We set $\rho^2 = \rho_1^2 + \rho_2^2$ and we consider the variables ρ_m along with $r_m = \rho_m^2$ $(m = 1, 2)$. Setting $r = \rho^2$, we get $r_1 + r_2 = r$. Following Kamenkov /9/, we introduce the new variables r and r_m' by the formulae $r_m = r r_m'$ $(m = 1, 2)$. The equations which determine ρ_m can be transformed to the form

$$\tfrac{1}{2} r^{\cdot} = \mu r(\alpha_1 r_1' + \alpha_2 r_2') + r^2 R_0 + \ldots \tag{5.4}$$
$$\tfrac{1}{2} r_1'^{\cdot} = \mu(\alpha_1 - \alpha_2) r_1' r_2' + r r_1' r_2' R_{1,2} + \ldots\ (1, 2)$$

and have the integral $r_1' + r_2' = 1$. The symbol (1, 2) means that the third equation in (5.4) can be obtained from the second equation by the transposition of indices $1 \leftrightarrow 2$. The variables r_m' in (5.4) vary over the interval $I = \{(r_1', r_2'): r_1' \geqslant 0,\ r_2' \geqslant 0,\ r_1' + r_2' = 1\}$. The following notation is used in (5.4):

$$R_0(r_1', r_2') = \varphi_{1,10}r_1'^2 + (\varphi_{1,01} + \varphi_{2,10})r_1'r_2' + \varphi_{2,01}r_2'^2 \tag{5.5}$$
$$R_{1,2}(r_1', r_2') = (\varphi_{1,10} - \varphi_{2,10})r_1' + (\varphi_{1,01} - \varphi_{2,01})r_2' \quad (1, 2)$$

$\varphi_{m,10}, \varphi_{m,01}$ $(m = 1, 2)$ are the coefficients of the continuous normal form in **y**. We shall write down the coefficients using the analytic arguments of /4/ (see also the article mentioned in the previous footnote). We set

$$G_{1,kl} = -(2k_1(s_{4k}^2s_{1l} + 2s_{1k}s_{4k}s_{4l})h_{2002} + 12s_{3k}^2s_{3l}h_{0040} +$$
$$2(s_{4k}^2s_{3l} + 2s_{3k}s_{4k}s_{4l})h_{0022} + 3k_1s_{4k}^2s_{4l}h_{1003}), \tag{5.6}$$
$$G_{2,kl} = -(2(s_{1k}^2s_{4l} + 2s_{1k}s_{4k}s_{1l})h_{2002} + 3(s_{4k}^2s_{1l} + 2s_{1k}s_{4k}s_{4l})h_{1003} +$$
$$2(s_{3k}^2s_{4l} + 2s_{3k}s_{4k}s_{3l})h_{0022} + 12s_{4k}^2s_{4l}h_{0004}),$$
$$G_{3,kl} = 2(s_{4k}^2s_{1l} + 2s_{1k}s_{4k}s_{4l})h_{2002} + 3s_{4k}^2s_{4l}h_{1003},$$
$$G_{kl} = \sum_{m=1}^{3} S_{lm}G_{m,kl} \quad (k, l = 1, \ldots, 4)$$

Here S_{lm} are the elements of the inverse matrix S^{-1}, that is, $S^{-1} = \| S_{lm} \|_{l,m=1}^{4}$. We get

$$8\varphi_{1,10} = G_{11} + G_{31} + G_{13} + G_{33}, \quad 4\varphi_{1,01} = G_{21} + \tag{5.7}$$
$$G_{41} + G_{23} + G_{43}, \quad 4\varphi_{2,10} = G_{12} + G_{32} +$$
$$G_{14} + G_{34}, \quad 8\varphi_{2,01} = G_{22} + G_{42} + G_{24} + G_{44}$$

Taking into account the known algebraic relations for S_{lm} $(l = 1, \ldots, 4,\ m = 1, 2, 3)$ and substituting (5.6), (5.2), (5.1), (2.5), and (2.6) into (5.7), we obtain the final expressions for the coefficients $\varphi_{m,10}, \varphi_{m,01}$ $(m = 1, 2)$ of the normal form in terms of the parameters **c** satisfying (5.3) and α_m, β_m $(m = 1, 2)$.

The following theorem on local uniform boundedness holds.

Theorem. Suppose that $|k_1 + k_2|$ and $|b_1 + a_2 - a_1|$ are small, the parameters **c** satisfy (5.3), and $e = \pm 1$. For such fixed **c** and $e = \pm 1$, let the coefficients $\varphi_{m,10}, \varphi_{m,01}$ $(m = 1, 2)$ given by (5.7) satisfy the following conditions: 1) if $\varphi_{1,01}\varphi_{2,10} \leqslant 0$ or $\varphi_{1,01} \leqslant 0$ and $\varphi_{2,10} \leqslant 0$ then the inequalities $\varphi_{1,10} < 0$ and $\varphi_{2,01} < 0$. hold; 2) if $\varphi_{1,01} > 0$ and $\varphi_{2,10} > 0$ then $\varphi_{1,10} < 0$, $\varphi_{2,01} < 0$, and $\varphi_{1,10}\varphi_{2,01} > \varphi_{1,01}\varphi_{2,10}$. Then the solutions of the equations of perturbed motions (2.7) are locally uniformly bounded with respect to the origin $p_m = q_m = 0$ $(m = 1, 2)$ of coordinates.

We will give an outline of the proof. The assumptions that $|k_1 + k_2|$ and $|b_1 + a_2 - a_1|$ are small and $\mathbf{c} \in G \cap (N_4 \cup N_2)$ ensure that two pairs of complex conjugate eigenvalues exist with small real parts, among which there are some positive ones. We consider the system (5.4) with (5.5) obtained from the equations of perturbed motions (2.7) with $\mathbf{c} \in G \setminus (R_2^0 \cup R_4^0)$. Let the coefficients (5.7) satisfy the restrictions imposed above. With the aid of the auxiliary Lyapunov type function proposed by Kamenkov /9/ it was shown /4/ (see also the article mentioned in the footnote) that under the conditions listed above the solutions of the normal form (5.4), (5.5) are locally uniformly bounded with respect to the equilibrium state $r = 0$. The expressions for the initial and current estimates q and p appearing in the definition of local uniform boundedness of the solutions of system (5.4), (5.5) with respect to the origin $r = 0$ are obtained in terms of the coefficients (5.7). The transformations of the variables of (2.7) which yield the normal form do not violate the property of local uniform boundedness. Consequently, the assertion of the theorem holds.

Remarks. 5.1. The zero solution $p_m = q_m = 0$ $(m = 1, 2)$ of the equations of perturbed motions (2.7) coincides with the state of equilibrium (2.2) of the reduced system.

5.2. The assumptions of the theorem contain certain restrictions expressed as inequalities for the coefficients (5.7) for the normal form. The expressions for the coefficients in terms of the parameters **c** satisfying (5.3), $e = \pm 1$, and α_m, β_m $(m = 1, 2)$ have been found above. It follows that the sufficient conditions for local uniform boundedness obtained in the theorem can be regarded as restrictions for the initial construction parameters of the gyroscope in gimbals.

The theorem on the continuous dependence of the solutions of (1.1) on the parameters and the assumptions of the theorem stated above imply the following result. The perturbed motions of a gyroscope in gimbals with dissipative and accelerating forces acting on the axes of the suspension rings are locally uniformly bounded in $p_\psi, p_\theta, p_\varphi, \psi, \theta$ with respect to the stationary motions (2.1) under parametric perturbations of the construction parameters of the system.

6. To provide an illustrative interpretation of the sufficient conditions for local uniform boundedness of the perturbed motions of a gyroscope in gimbals obtained above as restrictions upon c and $e=\pm 1$ we consider the special cases where the values of all the parameters except a and ω are fixed. To fix the parameters, we take into account Remarks 2.1 and 4.1. We set $a_1=2$, $b_1=1$, $c_1=3/2$, $a_2=1$, and $\delta=10^{-5}$.

We define the following special cases to be considered:

№	1°	2°	3°	4°	5°	6°
e	+1	+1	±1	±1	−1	−1
k_1	1.000	−1.005	2.000	−2.005	3.000	−3.005
k_2	−1.005	1.000	−2.005	2.000	−3.005	3.000

The inequalities for the coefficients (5.7) appearing among the assumptions of the theorem were verified using a computer. The analysis was carried out for the intersections of the rectangle $\{a\in[1/2;\,8]\}\,\{\omega\in[2;\,24]\}$ and the first regions G corresponding to the cases listed above.

For $e=1$, the computational results obtained in cases 3° and 4° are presented in Fig.1. In case 3° (4°) the curves $l_1\,(l_2)$ and R_2 are the boundaries of the domain $L_1\,(L_2)$ of local uniform boundedness. Some small neighbourhoods of the curves R_2 and R_4 are excluded from $L_1\,(L_2)$. Cases 1° and 2° are qualitatively the same as those in Fig.1. In case 6° the results of computations are presented in Fig.2. In cases 3° and 4° with $e=-1$ and in case 5° we failed to establish the domain of local uniform boundedness.

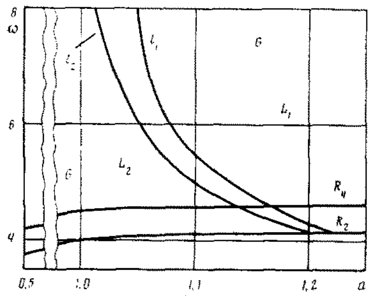

Fig.1

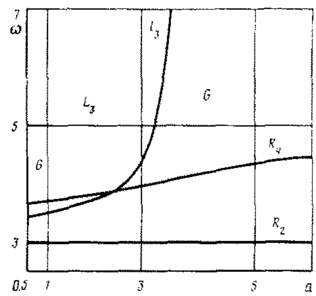

Fig.2

We mention that the initial and current estimates q and p of local uniform boundedness for the solutions of the normal form have been computed for a number of points in L_1, L_2 and L_3.

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